The conductivity of a partially ionized gas in alternating electric fields

By J. P. DOUGHERTY
Department of Applied Mathematics and Theoretical Physics, University of Cambridge

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A conductivity tensor is calculated for a partially ionized gas subject to electric fields which are harmonic in space and time, and to a uniform magnetic field. The collisions between charged particles and neutral molecules are included in the theory in an approximate way by means of a simplified form of Boltzmann's equation, as proposed by Bhatnagar, Gross and Krook. A simplified expression for the longitudinal component of the tensor is derived. Some applications of the results are mentioned, and will be described in detail in further papers.

1. Introduction

In many of the problems of plasma physics, the properties of the ionized medium are conveniently summarized by conductivity functions, σ, one for each of the species of charged particles. These enable any calculation to be regarded simply as the solution of Maxwell's equations together with Ohm's law. For instance, one often refers to the 'd.c. conductivity'; according to a crude calculation in which the collisions are represented by a simple drag force, this is $\sigma = Ne^2/m\nu$. Here N is the number density, e the charge, m the mass of the particles; ν is their collision frequency. The magnetoionic theory (Ratcliffe 1959) may be formulated in terms of a conductivity. This time it is an 'a.c. conductivity', and is a function of frequency; in the presence of an applied magnetic field it also acquires a tensor character. Even the theory of plasma oscillations, taking account of the thermal distribution of velocities, may be presented this way provided certain contours of integration are defined carefully so that Landau damping is not omitted. In this case the conductivity depends also on the wave number, $\mathbf{k}$. It was just this conductivity which was used to discuss incoherent scattering of radio waves (Dougherty & Farley 1960; Farley, Dougherty & Barron 1961). (The quantity there called the 'admittance', Y, is simply σ/e^2.)

The author and Dr D. T. Farley have recently encountered problems concerning partially ionized gases in which it is essential to include both the thermal motions and the collisions made by the charged particles with the neutral molecules. Examples include incoherent scattering in the lower ionosphere, and the stability of equatorial and auroral electrojets. In some cases, an external magnetic field has to be included also. A proper calculation of the conductivity would start with a Boltzmann's equation (Chapman & Cowling 1952) for the distribution function $f(\mathbf{x}, \mathbf{v}, t)$:

$$\frac{\partial f}{\partial t} + v_q \frac{\partial f}{\partial x_q} + \frac{1}{m} F_q \frac{\partial f}{\partial v_q} = \left[\frac{\partial f}{\partial t}\right]_c, \tag{1.1}$$

where $\mathbf{F}$ is the macroscopic force, and the term on the right represents the collision integrals, the other notation is as usual, and summation over q is assumed. This has already been done for the case of d.c. fields (constant in both space and time) by Cowling (1945). However, it appears to be very difficult to extend this work to include fields which oscillate rapidly in space and time. The standard method, due to Chapman and Enskog, for handling the Boltzmann equation, starts from the assumption that $[\partial f/\partial t]_c$ is the dominating term in (1.1) and proceeds by iteration. If the frequency of oscillation of the field approaches the collision frequency, or if the wavelength is as small as the mean free path, such an iteration is unjustified. On the other hand, we may not wish to regard the collisions as a perturbation on the collision-free calculations which have already been carried out.

To make progress in this intermediate case, the author has used a modified version of the simplified expression for $[\partial f/\partial t]_c$ which has often been proposed in the literature. In this paper, this equation is discussed, and conductivity tensors are calculated and simplified so that they are in a form ready for use in the applications. The latter will be discussed in separate papers.

During the final stages of preparation of this paper, the author's attention was drawn to the paper by Lewis & Keller (1962), in which a similar calculation is carried out, though by a somewhat different method. The results appear to be in agreement.

2. Approximate kinetic equation for a partially ionized gas

A number of writers (Kantrowitz & Petschek 1957; Bhatnagar, Gross & Krook 1954; Gross & Krook 1956; Deslodge & Matthysse 1960) have proposed that the simpler form

$$[\partial f/\partial t]_c = -\nu(f - f_{\max}) \tag{2.1}$$

be used for the right-hand side of (1.1). Here ν is a collision frequency and $f_{\max}$ is a suitable Maxwellian distribution. Although some justification (see for instance Deslodge & Matthysse 1960) for an equation of this type can be given, it is best regarded as empirical. It leads to a relatively simpler mathematical analysis than the proper collision integral, while having a number of characteristics in common with it. For example a uniform but non-Maxwellian distribution tends to turn into a Maxwellian one in a time of order ν^{-1}. Equation (2.1) may be regarded as having the same status in the kinetic theory of plasmas as has the simple 'friction' term in the magnetoionic theory, and in fact the two are closely related.

Care must be exercised in the choice of $f_{\max}$. In problems where small perturbations about a uniform Maxwellian distribution f_0 are considered, so that one writes $f = f_0 + f_1$, where f_1 is harmonic in space and time, the choice $f_{\max} = f_0$ suggests itself. In this case the collision-free analysis needs only the formal modification of replacing the angular frequency ω by $\omega - i\nu$, since the right-hand side of (2.1) is just $-\nu f_1$. For some purposes this is a good approximation, and has been suggested by various writers (Hagfors 1961; Renau, Camnitz & Flood 1961). But, as pointed out by Bhatnagar *et al.* (1954), this procedure is unsatisfactory because the kinetic equation (1.1) does not conserve particles locally, as may readily be seen by integrating over velocity space. The tendency for the plasma to relax towards a uniform Maxwellian distribution is thereby represented in an exaggerated way since the equation requires particles to disappear where there is an excess and reappear where there is a deficiency, in *physical* space as well as in velocity space.

Clearly $f_{\max}$ ought to be chosen as a Maxwellian distribution having the same total density in physical space as f itself rather than f_0. In that case

$$\int [\partial f/\partial t]_c \, d^3v = 0$$

is identically satisfied provided ν is independent of velocity, and the conservation of particles is ensured. Defining the number density in the usual way

$$N(\mathbf{x},t) = \int f(\mathbf{x},\mathbf{v},t)\,d^3v, \tag{2.2}$$

with similar definitions for N_0 and N_1 in terms of f_0 and f_1, we have

$$f_{\max}(\mathbf{x},\mathbf{v},t) = \frac{N(\mathbf{x},t)}{N_0} f_0(\mathbf{v}) = \left[1+\frac{N_1(\mathbf{x},t)}{N_0}\right] f_0(\mathbf{v}),$$

and (2.1) becomes

$$\left[\frac{\partial f}{\partial t}\right]_c = -\nu\left[f_1(\mathbf{x},\mathbf{v},t) - \frac{N_1(\mathbf{x},t)}{N_0} f_0(\mathbf{v})\right]. \tag{2.3}$$

This is equation (4) of Bhatnagar *et al.*'s paper. The final term is one which does not seem to have been included by other writers. It can in some circumstances be of the same order as the term $-\nu f_1$, and as we shall see in the applications, it can make successful a theory which would otherwise fail. However, it does make the analysis more complicated, as N_1 has to be expressed as an integral over f_1, so that Boltzmann's equation reassumes the character of an integral equation.

By taking moments of Boltzmann's equation in the standard way, the equations of momentum and energy can easily be constructed. It is found that the collision term does contribute to the rates of change of momentum and energy for each species of charged particle, but this is to be expected for a partially ionized gas as momentum and energy are being exchanged with the neutral molecules, the latter being assumed at rest. In fact one finds that the rate of loss of momentum is given by $m\nu\mathbf{u}$ per particle where $\mathbf{u}$ is the mean velocity for that species. This is a most desirable feature of our kinetic equation, since it is this expression that is used in the magnetoionic theory, which is entirely successful in dealing with the propagation of radio waves even if the collision frequency is comparable with the wave frequency. If, on the other hand, we were attempting to represent say electron-electron collisions by an equation such as (2.1), the modification of the equations of momentum and energy would be just as objectionable as that of the equation of continuity; this could be avoided by defining $f_{\max}$ to be a Maxwellian distribution with the same mean velocity, temperature and density as f, not just with the same density. In that case there would be still further terms in (2.3).

3. Formal expression for the conductivity tensor

To calculate the conductivity for a single species, we write Boltzmann's equation (1.1) explicitly, using (2.3)

$$\frac{\partial f}{\partial t}+v_q\frac{\partial f}{\partial x_q}+\frac{e}{m}\left[\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right]_q\frac{\partial f}{\partial v_q} = -\nu\left[f_1-\frac{N_1}{N_0}f_0\right], \tag{3.1}$$

where **E** and **B** are the electric and magnetic fields, and f_0 is the Maxwellian distribution

$$f_0(\mathbf{v}) = N_0\left[\frac{m}{2\pi KT}\right]^{\frac{3}{2}}\exp\left[-m\mathrm{v}^2/2KT\right]. \tag{3.2}$$

Following the usual method (Bernstein 1958; Farley *et al.* 1961) of linearization with the perturbations harmonic in space and time, we have $f=f_0+f_1$, $N = N_0+N_1$, $\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1$, where f_1, N_1, $\mathbf{B}_1$, and the whole of **E**, are small quantities with a factor

$$\exp\left[i(\omega t-\mathbf{k}.\mathbf{x})\right] \tag{3.3}$$

assumed, ω and **k** being constants. $\mathbf{B}_0$ is a uniform externally applied field. Since f_0 is isotropic, the term $(\mathbf{v}\times\mathbf{B})_q\,\partial f_0/\partial v_q$ is zero, and we have

$$i(\omega-i\nu-\mathbf{k}.\mathbf{v})f_1+\frac{e}{mc}(\mathbf{v}\times\mathbf{B}_0)_q\frac{\partial f_1}{\partial v_q} = -\frac{e}{m}E_q\frac{\partial f_0}{\partial v_q}+\nu\frac{N_1}{N_0}f_0. \tag{3.4}$$

Introduce the abbreviation

$$\mathscr{D}f_1 \tag{3.5}$$

for the left-hand side of this equation. $\mathscr{D}$ is in general a differential operator in velocity space, which also involves ω, ν, **k** and Ω, where

$$\Omega = eB_0/mc \tag{3.6}$$

is the gyro frequency. In the special case when $\mathbf{B}_0 = 0$, $\mathscr{D}$ is simply an algebraic factor. Then a formal solution of (3.5) is

$$f_1(v) = \mathscr{D}^{-1}\left[-\frac{e}{m}E_q\frac{\partial f_0}{\partial v_q}+\nu\frac{N_1}{N_0}f_0\right]. \tag{3.7}$$

The form taken by $\mathscr{D}^{-1}$ will be discussed later; for the present we note that it operates on f_0 and $\partial f_0/\partial v_i$, but it commutes with **E** or N_1 since these are constants so far as velocity space is concerned. Nevertheless, (3.7) is not by itself an explicit solution of (3.4) since we have

$$N_1 = \int f_1(\mathbf{v}')\,d^3v', \tag{3.8}$$

where the prime merely indicates a dummy variable of integration to be distinguished from **v** in (3.7). Integrating (3.7) over velocity, and relabelling **v** as **v**′, we find

$$N_1 = -\frac{e}{m}E_q\int\mathscr{D}^{-1}\frac{\partial f_0}{\partial v'_q}\,d^3v'+\frac{\nu N_1}{N_0}\int\mathscr{D}^{-1}f_0\,d^3v'.$$

Solving for N_1,

$$N_1 = \frac{-(e/m)\,E_q\int\mathscr{D}^{-1}(\partial f_0/\partial v'_q)\,d^3v'}{1-(\nu/N_0)\int\mathscr{D}^{-1}f_0\,d^3v'}, \tag{3.9}$$

a result which may now be substituted back into (3.7) to give an explicit expression for $f_1(\mathbf{v})$. We may now calculate the electric current density

$$\mathbf{j} = e\int f_1\,\mathbf{v}\,d^3v.$$

As this is linearly dependent on the components of **E** we may write

$$j_p = \sigma_{pq} E_q$$

and our conductivity tensor is (combining (3.7) and (3.9))

$$\sigma_{pq} = \frac{e^2}{KT}\left[\int v_p\, \mathscr{D}^{-1}(v_q f_0)\, d^3v + \frac{\nu\{\int \mathscr{D}^{-1}(v_q f_0)\, d^3v\}\{\int v_p\, \mathscr{D}^{-1} f_0\, d^3v\}}{N_0 - \nu \int \mathscr{D}^{-1} f_0\, d^3v}\right]. \quad (3.10)$$

Here we have used the relation $\partial f_0/\partial v_p = -mv_p f_0/KT$, satisfied by the Maxwell distribution. We note that $\mathscr{D}^{-1}$ does not in general commute with functions of the components of **v**, so that one must preserve the order of factors in the integrands. The prime introduced earlier has been discontinued as the integrations now appear quite separately.

We observe that the first term within the brackets in (3.10) is just what would be obtained from the collision-free results by the simple substitution $\omega \to \omega - i\nu$ as mentioned in § 2. The second, and more complicated, term arises from the modification of $f_{\max}$ required to preserve the equation of continuity.

It may be asked at this point whether the denominator of the fraction appearing in (3.10) can ever vanish; for if it can, we should infer that a new phenomenon of resonance has arisen from our collision term in Boltzmann's equation. This would clearly be surprising. In the case of a Maxwellian plasma it is readily shown that the denominator can vanish only for waves which decay in time. This means that any current present initially will decay if no electric field is applied, and it also ensures that all components of the conductivity satisfy the usual conditions (the Kramers–Kronig relations) expressing the principle of causality. The formal proof is given at the end of § 7.

To proceed with the evaluation of (3.10) we must first give expressions for $\mathscr{D}^{-1}g$, where g is a typical function of **v**. This is much easier for the case of no magnetic field than for non-zero field; so while the former case is contained in the latter, it is useful to devote a little space to the field-free case, in which the essential points appear more clearly.

4. Conductivity tensor in the absence of a magnetic field

When $\mathbf{B}_0 = 0$, a glance at (3.4) and (3.5) shows that the solution of an equation such as

$$\mathscr{D} f_1 = g(\mathbf{v})$$

is

$$f_1 = \mathscr{D}^{-1} g = \frac{g}{i(\omega - i\nu - \mathbf{k}.\mathbf{v})}. \quad (4.1)$$

We may then take $\mathbf{k} = (0, 0, k)$ without loss of generality, and consider all the integrals appearing in (3.10), using a Maxwellian f_0 as in (3.2). The integrations over v_x and v_y may be carried out at once, and one finds that σ_{pq} is diagonal, the remaining elements vanishing as f_0 is an even function. For the same reason, the second term of (3.10) vanishes in the case of the 'transverse' components σ_{xx} and σ_{yy}. These are therefore given by the simple procedure of modifying the frequency mentioned in the last section, starting from results given, for example, in Appendix I of Dougherty & Farley (1960). This is as one would expect: the last term of (3.10) arose from considering the equation of continuity, but this is irrelevant for purely transverse motions.

It remains to consider the longitudinal component, σ_{zz}. Here we introduce some dimensionless variables similar to those employed by Dougherty & Farley (1960) and Farley *et al.* (1961), by regarding $(2KT/m)^{\frac{1}{2}}$ as a typical velocity. Thus we define

$$\theta = \frac{\omega}{k}\left[\frac{m}{2KT}\right]^{\frac{1}{2}}, \quad \psi = \frac{\nu}{k}\left[\frac{m}{2KT}\right]^{\frac{1}{2}}, \quad \eta = v_z\left[\frac{m}{2KT}\right]^{\frac{1}{2}}, \tag{4.2}$$

as normalized frequency, collision frequency and particle velocity, and a normalized conductivity

$$\mathbf{y} = (KTk^2/N_0 e^2\omega)\,\boldsymbol{\sigma}. \tag{4.3}$$

In these terms the longitudinal component of (3.10) becomes

$$y_{zz} = -\frac{i}{\pi^{\frac{1}{2}}\theta}\int\frac{\eta^2 e^{-\eta^2}\,d\eta}{\theta - i\psi - \eta} - \frac{\psi}{\theta}\left[\frac{1}{\pi^{\frac{1}{2}}}\int\frac{\eta\, e^{-\eta^2}\,d\eta}{\theta - i\psi - \eta}\right]^2\left[1 + \frac{i\psi}{\theta}\int\frac{e^{-\eta^2}\,d\eta}{\theta - i\psi - \eta}\right]^{-1}. \tag{4.4}$$

The range of integration in each of these integrals is of course $[-\infty, \infty]$, but we must follow the Landau procedure in the choice of contour in the complex η-plane. With the present sign convention, this means that the contour is the real axis if $\mathscr{I}(\theta - i\psi) < 0$, but is indented upwards if $\mathscr{I}(\theta - i\psi) > 0$, so as to lie above the singularity $\eta = \theta - i\psi$. In other words, the contour is the real axis except for waves which decay faster than $e^{-\nu t}$.

Further simplification of (4.4) is achieved by defining a function

$$G(\zeta) \equiv \pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{e^{-\eta^2}\,d\eta}{\zeta - \eta} \equiv 2e^{-\zeta^2}\int_{-i\infty}^{\zeta} e^{p^2}\,dp, \tag{4.5}$$

where the path of integration for the first integral obeys the convention just described. This function is just $Z(-\zeta)$ where Z is the function tabulated by Fried & Conte (1961). It is readily shown that

$$\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{\eta\, e^{-\eta^2}\,d\eta}{\zeta - \eta} \equiv \zeta G(\zeta) - 1,$$

and

$$\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{\eta^2 e^{-\eta^2}\,d\eta}{\zeta - \eta} \equiv \zeta^2 G(\zeta) - \zeta,$$

so that all the integrals in (4.4) can be put in terms of G. On doing this, the expression for y_{zz} finally reduces to

$$y_{zz}(\theta, \psi) = -\frac{(\theta - i\psi)\,G(\theta - i\psi) - 1}{i - \psi\, G(\theta - i\psi)}. \tag{4.6}$$

We note that the apparent singularity in y_{zz} at $\theta = 0$ (see (4.4)) disappears on account of the cancellation of some of the terms, a point which becomes significant in the applications.

5. Conductivity tensor in the presence of a magnetic field

When $B_0 \neq 0$, the solution of an equation such as

$$\mathscr{D} f_1 = g(\mathbf{v}), \tag{5.1}$$

as required by (3.10) becomes much more difficult. Fortunately it has essentially been done already, by Bernstein (1958) and others. Our present requirements merely amount to new choices of $g(\mathbf{v})$ and the modification of ω to become $\omega - i\nu$. We shall find it convenient to adopt the method and notation of our earlier derivation for the case of a collision-free plasma in a magnetic field (Farley *et al.* 1961, Appendix). There the operator $\mathscr{D}$ was regarded as a time derivative following the unperturbed orbit of a particle in phase space. We can do the same here; it being realized that 'unperturbed orbit' continues to mean the orbit of a particle in the field $\mathbf{B}_0$ *without* collisions, as the collisions have been included in (5.1) by the modification of ω. The earlier derivation gives at once

$$\mathscr{D}^{-1} g(\mathbf{v}) = \int_0^\infty g(\mathbf{v}^{(t)}) \exp\left[-i(\omega - i\nu)t + i\mathbf{v}.\mathbf{p}(t)\right] dt. \tag{5.2}$$

Here $\mathbf{v}^{(t)}$ is the velocity of a particle at a time t *earlier* than when it has a specified velocity $\mathbf{v}$, assuming that it travels along the unperturbed orbit and $\mathbf{p}(t)$ is a certain vector characteristic of the unperturbed orbit and also of $\mathbf{k}$. The solution of our equation may be regarded as an accumulation of perturbations over the previous history of the particle and the exponential factor is a transfer function (or integrating factor) representing the influence of conditions at time t earlier. Equivalently, the exponential factor may be regarded as a Green's function and the unperturbed orbits are characteristic curves of the differential equations. We have tacitly assumed that no perturbations are present in the infinite past, but this is indeed the case if the perturbing field is switched on adiabatically and that is known to be equivalent to Landau's rule for determining the path of integration in the case $\mathbf{B}_0 = 0$ (see § 4).

The connexion between this solution of (5.1) and Bernstein's (1958) integrating factor is shown explicitly by Renau *et al.* (1961).

To give actual expressions for $\mathbf{v}^{(t)}$, $\mathbf{p}(t)$, and to handle concisely the vector or tensor character of the functions which, according to (3.10), play the part of g, it is convenient to introduce 'circularly polarized' co-ordinates, as in Farley *et al.* (1961). These are (x_1, x_0, x_{-1}) defined by

$$x_{\pm 1} = (x \pm iy)/2^{\frac{1}{2}}, \quad x_0 = z, \tag{5.3}$$

where (x, y, z) for the purposes of this definition are Cartesian axes with O_z parallel to B_0. The suffices $1, 0, -1$ will always be denoted by Greek letters, and the summation convention does not apply to them. For $\gamma = 0$, formulae are to be interpreted by taking the limit $\gamma \to 0$ where necessary. Then quoting again from the earlier paper,

$$v_\gamma^{(t)} = v_\gamma e^{i\gamma\Omega t}, \tag{5.4}$$

$$p_\gamma(t) = k_\gamma[e^{-i\gamma\Omega t} - 1]/(-i\gamma\Omega), \tag{5.5}$$

Ω being the gyro-frequency eB_0/mc. We may, without loss of generality, choose $k_1 = k_{-1} = k\sin\alpha/2^{\frac{1}{2}}, k_0 = k\cos\alpha$, α being the angle between $\mathbf{k}$ and $\mathbf{B}_0$.

It must also be noted that the scalar product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is $\sum_\gamma A_{-\gamma}B_\gamma$ in these co-ordinates. We therefore obtain the tensor component $\sigma_{\lambda\mu}$ of (3.10) by replacing v_p, v_q by $v_\lambda, v_{-\mu}$. We now apply (5.2) in turn to the various integrals required in (3.10).

(*a*)
$$\int \mathscr{D}^{-1}f_0\,d^3v = \int_{\mathbf{v}}\int_0^\infty f_0(\mathbf{v})\exp\left[-i(\omega-i\nu)t+i\mathbf{p}.\mathbf{v}\right]dt\,d^3v$$

after noting that $\mathbf{v}^{(t)}$ and $\mathbf{v}$ have the same amplitude so that f_0 is the same at each. But for the Maxwellian distribution

$$\int_{\mathbf{v}} f_0\,e^{i\mathbf{p}.\mathbf{v}}\,d^3v = N_0\,e^{-\mathbf{p}^2KT/2m}, \tag{5.6}$$

so that
$$\int \mathscr{D}^{-1}f_0\,d^3v = N_0\int_0^\infty \exp\left[-i(\omega-i\nu)t-\mathbf{p}^2KT/2m\right]dt, \tag{5.7}$$

and we note
$$\mathbf{p}^2 = 2k^2\Omega^{-2}(1-\cos\Omega t)\sin^2\alpha + t^2k^2\cos^2\alpha. \tag{5.8}$$

(*b*)
$$\int v_\lambda\,\mathscr{D}^{-1}f_0\,d^3v = \int_{\mathbf{v}}\int_0^\infty v_\lambda f_0(\mathbf{v})\exp\left[-i(\omega-i\nu)t+i\mathbf{p}.\mathbf{v}\right]d^3v\,dt.$$

But
$$\int_{\mathbf{v}} v_\lambda\,f_0(\mathbf{v})\,e^{i\mathbf{p}.\mathbf{v}}\,d^3v = -i\frac{\partial}{\partial p_{-\lambda}}\int f_0\,e^{i\mathbf{p}.\mathbf{v}}\,d^3v,$$
$$= -iN_0\frac{\partial}{\partial p_{-\lambda}}\left[e^{-\mathbf{p}^2KT/2m}\right], \quad \text{(using (5.6) again)}$$
$$= \frac{iN_0KT}{m}\,p_\lambda\,e^{-\mathbf{p}^2KT/2m}.$$

So
$$\int v_\lambda\,\mathscr{D}^{-1}f_0\,d^3v = (iN_0KT/m)\int_0^\infty p_\lambda\exp\left[-i(\omega-i\nu)t-\mathbf{p}^2KT/2m\right]dt. \tag{5.9}$$

(*c*) For
$$\int\mathscr{D}^{-1}(v_{-\mu}f_0)\,d^3v$$

a similar procedure applies; the only difference resulting from the presence of $v_{-\mu}$ *after* the operator $\mathscr{D}^{-1}$ instead of before it, is that we need to evaluate $\mathscr{D}^{-1}v^{(t)}_{-\mu}f_0(\mathbf{v}) = \mathscr{D}^{-1}v_{-\mu}\,e^{-i\mu\Omega t}f_0(\mathbf{v})$, so that a factor $e^{-i\mu\Omega t}$ is introduced. Hence

$$\int\mathscr{D}^{-1}(v_{-\mu}f_0)\,d^3v = \frac{iN_0KT}{m}\int_0^\infty p_{-\mu}\exp\left[-i(\omega-i\nu+\mu\Omega)t-\mathbf{p}^2KT/2m\right]dt. \tag{5.10}$$

(*d*) Finally, we have

$$\int v_\lambda\,\mathscr{D}^{-1}(v_{-\mu}f_0)\,d^3v$$
$$= \int_{\mathbf{v}}\int_0^\infty v_\lambda\,v_{-\mu}\,f_0(\mathbf{v})\exp\left[-i(\omega-i\nu+\mu\Omega)t+i\mathbf{p}.\mathbf{v}\right]d^3v\,dt,$$
$$= \frac{N_0KT}{m}\int_0^\infty\left(\delta_{\lambda\mu}-\frac{KT}{m}p_\lambda p_{-\mu}\right)\exp\left[-i(\omega-i\nu+\mu\Omega)t-\mathbf{p}^2KT/2m\right]dt, \tag{5.11}$$

using (5.6) as before, this time involving second derivatives with respect to components of $\mathbf{p}$.

Equations (3.10), (5.5) and (5.7)–(5.11) summarize the means by which $\sigma_{\lambda\mu}$ may be calculated in these co-ordinates; the general result is of formidable complexity. Fortunately, the applications considered so far call for less information than is contained in these equations, and considerable simplification is available.

6. Zero temperature limit

Here, we ask whether our formulae reduce to the familiar a.c. conductivity used in magnetoionic theory, even if a magnetic field and collisions are included. We allow T to tend to zero, so that f_0 becomes a delta function. Equation (3.10) shows that the large bracket should contain a part proportional to T, together with terms involving higher powers of T which may be neglected in the limit. The first term in the bracket is just the integral evaluated in (5.11) and so after cancelling KT, we have a contribution

$$\sigma_{\lambda\mu} = \frac{N_0 e^2}{m}\int_0^\infty \exp[-i(\omega - i\nu + \mu\Omega)]\,\delta_{\lambda\mu}\,dt = \frac{N_0 e^2}{mi(\omega - i\nu + \mu\Omega)}\,\delta_{\lambda\mu}. \tag{6.1}$$

The last step is valid only if $\mathscr{I}(\omega - i\nu) < 0$, but as we remarked earlier, the Landau prescription requires us to calculate $\boldsymbol{\sigma}$ for this domain first and then extend the result by analytic continuation. Hence (6.1) is valid for all the complex ω-plane.

The remainder of (3.10) gives no contribution as each of the integrals appearing in the numerator of the fraction is proportional to T, so giving a term proportional to T^2 (see (5·9)–(5.10)). So in this limit, the final term of (2.3), which we added to maintain the equation of continuity, leads in the end to no contribution and (6.1) is the complete expression. We note that it is diagonal.

It is readily verified that (6.1) is indeed the conductivity of a cold partially ionized gas. For if $\mathbf{u}$ is the (Lagrangian) velocity vector for the species under consideration, the equation of motion is

$$m\dot{\mathbf{u}} = -m\nu\mathbf{u} + e\left(\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B}_0\right)$$

to the first order; assuming time dependence $\exp(i\omega t)$ and translating to the $(1, 0, -1)$ co-ordinates this becomes

$$u_\lambda(i\omega + \nu + i\lambda\Omega) = \frac{e}{m}E_\lambda.$$

Hence
$$j_\lambda = N_0 e u_\lambda = \frac{N_0 e^2}{mi(\omega - i\nu + \lambda\Omega)}E_\lambda.$$

We thus have a diagonal conductivity tensor which is in fact the same as (6.1), on account of the properties of the Kronecker δ. Therefore if the limit $T \to 0$ is taken, the general results must reduce to those of the familiar magneto-ionic theory, or its generalizations if more than one species is included.

7. The longitudinal conductivity

By the longitudinal component of $\boldsymbol{\sigma}$ we mean σ_{zz} evaluated in Cartesian axes for which $\mathbf{k}$ is along the z-axis. In terms of our present axes it is therefore

$$\sigma_{zz} = k^{-2} \sum k_{-\lambda} k_{\mu} \sigma_{\lambda\mu}, \tag{7.1}$$

and is the analogue of the quantity calculated in § 4 for no magnetic field. There are several applications for which the full Maxwell equations can be replaced by simple electrostatics to very good approximation in which case only σ_{zz} is relevant, as described, for example by Farley *et al.* (1961).

Referring to equation (3.10), we observe that we shall need to evaluate

$$\sum_{\lambda} k_{-\lambda} \int v_{\lambda} \mathscr{D}^{-1} f_0 \, d^3v, \quad \sum_{\mu} k_{\mu} \int \mathscr{D}^{-1}(v_{-\mu} f_0) \, d^3v \quad \text{and} \quad \sum_{\lambda,\mu} k_{-\lambda} k_{\mu} \int v_{\lambda} \mathscr{D}^{-1}(v_{-\mu} f_0) \, d^3v. \tag{7.2}$$

For the first of these, we use (5.9) and the fact that

$$\sum_{\lambda} k_{-\lambda} p_{\lambda} = \mathbf{k}.\mathbf{p} = k^2 \left[\sin^2\alpha \frac{\sin \Omega t}{\Omega} + t \cos^2\alpha \right] \quad \text{(using (5.5))},$$

$$= \frac{d}{dt}(\tfrac{1}{2}\mathbf{p}^2) \quad \text{(using (5.8))}.$$

For the second, we use (5.10), and

$$\sum_{\mu} k_{\mu} p_{-\mu} e^{-i\mu\Omega t} = \sum_{\mu} k_{\mu} k_{-\mu} \frac{e^{i\mu\Omega t} - 1}{i\mu\Omega} e^{-i\mu\Omega t},$$

$$= \sum_{\mu} p_{\mu} k_{-\mu} = \mathbf{k}.\mathbf{p} = \frac{d}{dt}(\tfrac{1}{2}\mathbf{p}^2).$$

Hence the first two expressions in (7.2) are in fact both equal to

$$\frac{iN_0 KT}{m} \int_0^{\infty} \frac{d}{dt}(\tfrac{1}{2}\mathbf{p}^2) \exp[-i(\omega - i\nu)t - \mathbf{p}^2 KT/2m] \, dt$$

$$= -iN_0 \int_0^{\infty} \exp[-i(\omega - i\nu)t] \frac{d}{dt}[\exp(-\mathbf{p}^2 KT/2m)] \, dt,$$

$$= iN_0 + (\omega - i\nu) N_0 \int_0^{\infty} \exp[-i(\omega - i\nu)t - \mathbf{p}^2 KT/2m] \, dt, \tag{7.3}$$

the last step involving an integration by parts, using $\mathbf{p}(0) = 0$.

The third expression in (7.2) can be treated similarly, starting from (5.11) and involving two integrations by parts (a more cumbersome derivation of this last result appears at the end of the appendix of the paper by Farley *et al.* 1961); the result is

$$N_0(\omega - i\nu)[i + (\omega - i\nu) I], \tag{7.4}$$

where I is the integral appearing in (7.3), so we may also rewrite (5.7) as

$$\int \mathscr{D}^{-1} f_0 \, d^3v = N_0 I. \tag{7 5}$$

On combining equations (3.10), (7.1)–(7.5) and noting that some terms cancel in a way analogous to the step from (4.4) to (4.6), we find

$$\sigma_{zz} = \frac{N_0 e^2}{KTk^2} \frac{[i + (\omega - i\nu) I] \, \omega}{1 - \nu I}. \tag{7.6}$$

On making the same normalizations as for no magnetic field, namely equations (4.2) and (4.3), with the addition of a normalized gyrofrequency

$$\phi = \frac{\Omega}{k}\left[\frac{m}{2KT}\right]^{\frac{1}{2}}, \tag{7.7}$$

we find

$$y_{zz} = \frac{i+(\theta-i\psi)J}{1-\psi J}, \tag{7.8}$$

where J is the 'normalized Gordeyev Integral'

$$J(\theta-i\psi,\phi,\alpha) \equiv \int_0^\infty \exp\left[-i(\theta-i\psi)t-\phi^{-2}\sin^2\alpha\,\sin^2\left(\tfrac{1}{2}\phi t\right)-\tfrac{1}{4}t^2\cos^2\alpha\right]dt, \tag{7.9}$$

so J has the same meaning as in Farley *et al.* (1961), equation (7.2).

Comparing with (4.6) we see that iJ plays the same role as G in the case of no field, and it is readily shown that the two are indeed equal in the limit $\phi \to 0$.

Equation (7.8) is the final form of our result, and has been found to be a convenient starting point for two quite separate applications concerned with ionospheric physics; these are reported in separate papers (Dougherty & Farley 1963; Farley 1963*b*) on incoherent scatter and electrojet stability respectively. A letter (Farley 1963*a*) summarizing the latter has already appeared.

It is now easy to prove that y_{zz} has no singularity for $\mathscr{I}(\omega) < 0$. Since J is an analytic function, the only question is whether the denominator of (7.8), $1-\psi J$, can vanish when ω, and therefore θ, is in the lower half plane. As this would be synonymous with the vanishing of the denominator in (3.10), we shall in fact establish that all components of $\boldsymbol{\sigma}$ are analytic in the lower half plane, so that any singularities correspond only to decaying waves, a property we quoted without proof in § 3.

If $\mathscr{I}(\theta) < 0$, $\theta - i\psi$ may be written $\xi - i\eta$, where ξ and η are real and $\eta > \psi > 0$. Using (7·9) we have

$$\mathscr{R}(J) = \int_0^\infty \cos\xi t \exp\left[-\eta t-\phi^{-2}\sin^2\alpha\,\sin^2\tfrac{1}{2}\phi t-\tfrac{1}{4}t^2\cos^2\alpha\right]dt.$$

Hence

$$|\mathscr{R}(J)| \leqslant \int_0^\infty \exp\left[-\eta t-\phi^{-2}\sin^2\alpha\,\sin^2\tfrac{1}{2}\phi t-\tfrac{1}{4}t^2\cos^2\alpha\right]dt$$

$$< \int_0^\infty e^{-\eta t}\,dt = 1/\eta.$$

So

$$|\psi\mathscr{R}(J)| \leqslant \psi/\eta < 1,$$

and $\psi J = 1$ is impossible.

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